SFB 823

Optimal designs for additional day effects in generalized linear models with gamma distributed response

Discussion Paper

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Nr. 5/2013

Contents

1. Introduction

In planned experiments the choice of an efficient experimental design is a vital question. We consider a specific situation where the experiments are conducted at two different days. A generalized linear model is estimated on the basis of the available data from the first day. In a second step day effects are to be added to the model from a limited number of additional experiments and we are interested in an optimal design of experiment for the necessary additional experimental runs. This question arises in an application to thermal spraying where the process is highly influenced by latent day specified effects and generalized linear models turn out to be a suitable class of models.

Generalized linear models provide models for situations in which the response is not necessarily normal, but follows a distribution from any exponential family where the mean is modeled as a function of the predictor. Unlike the linear regression case, optimal designs then may depend on the unknown parameter value as well as the specifically chosen model components. So far, optimal designs for this situation are rarely treated in the literature and if they are mostly with an emphasis on binary or Poisson response variables. Khuri et al. (2006) give a very nice review of the most common approaches to handle the so-called design dependency problem, namely locally optimal designs, sequential designs, Bayesian designs and quantile dispersion graphs. Woods et al. (2006) develop a "compromise" design selection criterion that takes uncertainties in the parameters as well as in the link function and the predictor into account by averaging over a chosen parameter and model space. With regard to this generation of "compromise" designs Dror and Steinberg (2006) present a heuristic using K-means clustering over local D-optimal designs that is robust against the mentioned uncertainties.

The design problem investigated in this paper differs from the problems discussed in the literature in several perspectives. Firstly, the response in the thermal spraying process is multivariate, while the literature usually discusses designs for a univariate response. Secondly, we investigate the situation where a part of the data has been already observed on an initial day and a design is required for collecting additional data on any current day, which has good properties to estimate a likely day-effect, describing the difference in the spraying between two days.

Hence, model selection for each component of the response can be performed on the basis of the initial design, but a compromise design has to be found for the models corresponding to the different components of the response, which additionally addresses the problem of uncertainty with respect to the mode parameters. For the purpose of detecting differences between days the D-optimality criterion might not be appropriate and we also consider alternative criteria designed for model discrimination.

The remaining part of the paper is organized as follows. In Section 2 we give an introduction to the problem of thermal spraying and motivate the application of generalized linear models (GLM) in this context. For the sake of transparency, we concentrate on Gamma-distributed responses and avoid most of the general notation of GLM. Section 4 is devoted to optimal design problems and we discuss locally, multi-objective or compromise designs and optimal designs for identifying an additional day effect. In Section 5 we return to the problem of designing additional experiments for the thermal spraying problem. In particular, we demonstrate that a reference design can be substantially improved with respect to its efficiency of estimating all parameters while moderate improvements can be achieved for testing for an additional day effect. Finally all optimal designs and additional material are presented in an entire Appendix.

2. Statistical modeling of thermal spraying

Thermal spraying technology is widely used in industry to apply coatings on surfaces, aiming e.g. at better wear protection or durable medical instruments. However, due to uncontrollable factors thermal spraying processes are often lacking in reproducibility, especially if the same process is repeated on different days. Furthermore an immediate analysis of the coating quality is usually not feasible as it requires time and results in destruction. A solution to this problem possibly lies in measuring properties of particles in flight based on the assumption that they carry the needed information of uncontrollable day effects [Tillmann et al. (2010)]

As application a HVOF (high-velocity oxygen-fuel spray) spraying process is regarded where WC-Co powder is melted and at high-speed applied to a surface by a

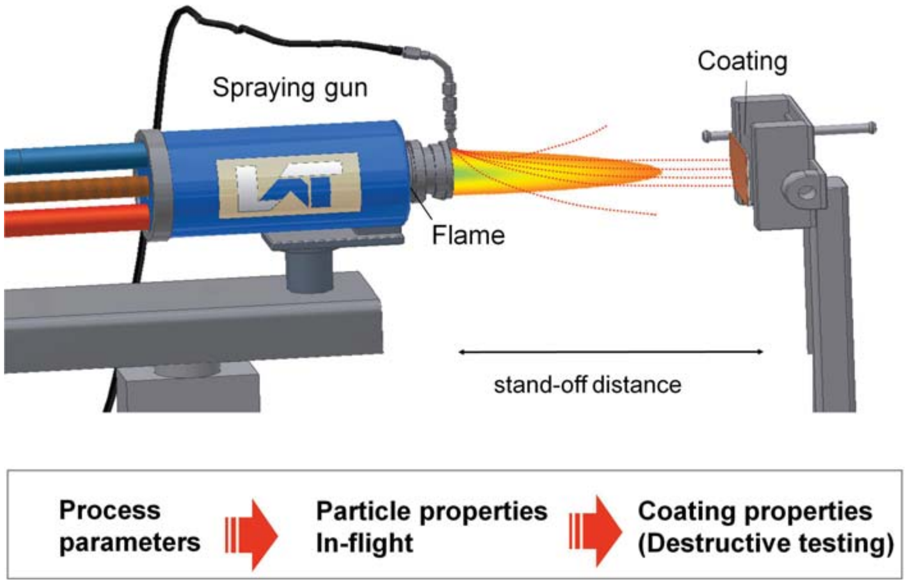

Figure 2.1: *Thermal spraying process*

spraying gun. Of interest is the influence of process parameters on in-flight properties of the coating powder. Figure 2.1 depicts the thermal spraying process. Preliminary screening experiments [Tillmann et al. (2010)] identify four relevant process parameters: The amount of kerosine (K) in liter per hour used, the ratio lambda of kerosine to oxygen (L) and the feeder disc velocity (FDV) as well as the stand-off-distance (D). The last parameter describes the distance from the spraying gun to the component which is coated and thereby also to the device measuring properties of the particles in flight. The device measures the temperature and velocity of properties in flight as well as flame width and flame intensity. The considered process parameters and in-flight properties are summarized in Table 2.1.

process parameters	in-flight properties
stand-off-distance (D)	temperature
amount of kerosine (K)	velocity
ratio of kerosine to oxygen (L)	flame width
feeder disc velocity (FDV)	flame intensity

Table 2.1: *Process parameters and in-flight properties*

Summary statistics of the in-flight measurements provide responses which have successfully been modeled by generalized linear models with Gamma distribution and different link functions based on central composite designs [Tillmann et al. (2012); Rehage et al. (2012)]. To capture the effect of unobservable day specific influences, e.g. created by room temperature and moisture, day effects have been added to the linear predictor of the models [Tillmann et al. (2012); Rehage et al. (2012)]. These effects have to be estimated from few additional experiments on any current day. It is therefore of high interest to determine optimal experimental designs for this specific task.

3. Measuring information in generalized linear models

In this section we give some background on the generalized linear models which are used to model the thermal spraying process. As common in statistical literature, we denote the real valued response by Y and the predictor by a q-dimensional variable x. In the application Y presents either the temperature, velocity, flame width or the flame intensity, while the predictor is a four-dimensional variable containing the machine parameters stand-off-distance, amount of kerosine, ratio of kerosine to oxygen and feeder disc velocity.

3.1. Gamma distributed responses

Let (Y_i, x_i), $i = 1, \cdots, n$, be a sample of observations where $x_i = (x_{1i}, \cdots, x_{qi})^T \in \mathbb{R}^q$ are explanatory variables and $Y_i \in \mathbb{R}$ is the response at experimental condition x_i $(i = 1, \dots, n)$. In contrast to linear models the response modeled by a generalized linear model may follow a distribution from the exponential family. Tillmann et al. (2012) and Rehage et al. (2012) showed that the in-flight properties in the thermal spraying application can be adequately modeled by generalized linear models with Gamma distributed response. These models are defined by the density

$$f(y|x, \beta) = \frac{1}{\Gamma(\nu)} \left(\frac{\nu}{\mu}\right)^{\nu} y^{\nu-1} e^{-\frac{\nu}{\mu}y}, \quad y \geq 0,$$

and mean

$$\mu = E(Y|x) = g^{-1}\left(z^T\beta\right) \tag{1}$$

where $g(\cdot)$ is an appropriate (known) link function, $z = z(x) \in \mathbb{R}^p$ is a vector of regression functions depending on the explanatory variables x, $\beta \in \mathbb{R}^p$ denotes an unknown parameter vector and $\mu > 0$ and $\nu > 0$ denote the mean and shape parameter, respectively [Fahrmeir and Tutz (2001)]. Common link functions for the Gamma distribution include the identity $g(\mu) = \mu$, the canonical link $g(\mu) = -1/\mu$ and the log link $g(\mu) = \log(\mu)$. For the first two link functions restrictions regarding β have to be made such that the conditional expectation μ is non-negative.

If n independent observations at experimental conditions $x_1, \ldots, x_n$ are available and the inverse of the link function g^{-1} is twice continuously differentiable, it follows by a straightforward calculation that the Fisher information matrix for the parameter β is given by

$$I(\beta) = \nu^2 \sum_{i=1}^{n} w(z_i^T \beta) z_i z_i^T, \tag{2}$$

where the weight function is defined by

$$w(\mu) = ((\log g^{-1}(\mu))')^2 = \frac{1}{(g'(g^{-1}(\mu))g^{-1}(\mu))^2}.$$

The covariance matrix of the maximum likelihood estimate for the parameter β can be approximated by the inverse of the information matrix $I(\beta)$. Note that for the different link functions the corresponding information matrices differ only with respect to the weight $w(\mu)$, and the weights corresponding to the Gamma distribution for the named link functions are shown in Table 3.1.

Link function $g(\cdot)$	weight in (2)
$g(\mu) = \mu$	$1/(z_i^T\beta)^2$
$g(\mu) = 1/\mu$	$1/(z_i^T\beta)^2$
$g(\mu) = \log(\mu)$	1

Table 3.1: *Weights in the information matrix* (2) *for the Gamma distribution with identity, canonical and log link*

In each case the information matrix depends on the sample size n, the link function g, the vector of regression functions $z(x)$ and especially on the parameter β. Throughout this paper we consider a quadratic response function for $g(E[Y|x])$, that is

$$z^T\beta = \beta_0 + \sum_{i=1}^{q} \beta_i x_i + \sum_{i=1}^{q} \sum_{j=q+1}^{q} \beta_{ij} x_i x_j. \tag{3}$$

4. Optimal designs for generalized linear models

Optimal designs maximize a functional, say Φ, of the Fisher information matrix with respect to the choice of the experimental conditions $x_1, \ldots, x_n$, and numerous criteria have been proposed in the literature to discriminate between competing designs [see Pukelsheim (2006)]. The commonly used optimality criteria (such as the D-, A- or E-optimality criterion) are positively homogenous, that is $\Phi(\lambda I(\beta)) = \lambda \Phi(I(\beta))$ whenever $\lambda \geq 0$ [see Pukelsheim (2006)]. Consequently, an optimal design maximizing a functional of the Fisher information matrix will not depend on the parameter ν, but it will depend on the parameter β. Therefore these designs are called locally optimal designs and were at first discussed by Chernoff (1953). Since this fundamental paper numerous authors have worked in the construction of locally optimal designs. We refer to some recent work in this direction by Yang and Stufken (2009), Yang (2010) and Dette and Melas (2011), who discuss admissible classes of locally optimal designs for nonlinear regression models with a one-dimensional predictor.

In situations where preliminary knowledge regarding the unknown parameters of a generalized linear model is available, the application of locally optimal designs is well justified. A typical example are phase II dose finding trials, where some useful information is already available from phase I [see Dette et al. (2008)]. A further situation was described in the introduction. Here a couple of experiments were already performed on the basis of a central composite design, and 8 new experiments have to be planned for further investigations. On the basis of the available observations parameter estimates and standard deviations are available, which can be used in the corresponding local optimality criteria. Locally D-optimal designs will be discussed in Section 4.1.

On the other hand, locally optimal designs are often used as benchmarks for commonly proposed designs (see also the discussion in Section 5). Moreover, they are the basis for more sophisticated design strategies, which require less precise knowledge about the model parameters, such as sequential, Bayesian or standardized maximin optimality criteria [see Pronzato and Walter (1985), Chaloner and Verdinelli (1995) and Dette (1997) among others]. Optimal designs with respect to the latter criteria are called robust designs and will be discussed in Section 4.2.

4.1. Locally D-Optimal designs

As Myers et al. (2002) point out, the D-optimality criterion is a commonly used design selection criterion especially for industrial experiments. To be precise, consider a link function g and a regression model of the form (3) defined with corresponding vector $z = z(x)$ and parameter β. We collect the model information in the vector $s = (g, z, \beta)$. In order to reflect the dependency of the Fisher information matrix in (2) on a particular model specified by the link function g and corresponding parameter β we introduce the notation

$$I(\mathbf{X}, s) = \sum_{i=1}^{n} w(z_i, \beta) z_i z_i^{'} \tag{4}$$

for the Fisher information matrix, where $\mathbf{X} = (x_1, \cdots, x_n)$ denotes the design and $z_i = z(x_i)$ $(i = 1, \ldots, n)$. Following Chernoff (1953) we call a design $\mathbf{X}_s^*$ locally D-optimal if it maximizes the determinant of the Fisher information matrix

$$\Phi_D(\mathbf{X}, s) = |I(\mathbf{X}, s)| \, . \tag{5}$$

Note that the locally D-optimal design depends on the link function g, the model z and the corresponding unknown parameter vector β, which justifies our notation $\mathbf{X}_s^*$ $(s = (g, z, \beta))$. Since this fundamental paper numerous authors have worked in the construction of locally D-optimal designs, where it is usually assumed that information regarding the unknown parameter in a specific fixed model is available [see for example Ford et al. (1992), Biedermann et al. (2006b), Fang and Hedayat (2008), Dette et al. (2010) among many others]. The locally D-criterion (and other optimal designs with respect to locally optimality criteria) have been criticized because of its dependences on the specific choice of the parameter β. However, there are numerous situations where preliminary knowledge regarding the unknown parameters is available, such that the application of locally optimal designs is well justified (see the discussion at the beginning of this section). A further common criticism of the criterion (5) is that it requires the specification of the model and the link function and there are several situations where a design for specific model is not efficient for an alternative competing model [see Dette et al. (2008)]. In the following sections we briefly discuss different approaches to find D-optimal designs which are less sensitive with respect to a misspecification of link, model and the parameter vector β.

4.2. Multi-objective designs

The problem of addressing model uncertainty (with respect to the form of the regression function or prior information regarding the unknown parameter) has a long history. Läuter (1974a) proposed a criterion which is based on a product of the determinants of the information matrices in the various models under consideration and yields designs which are efficient for a class of given models. Lau and Studden (1985) and Dette (1990) explicitly determined optimal designs with respect to Läuter's criterion for a class of trigonometric and polynomial regression models, respectively. In the case where the form of the model is fixed and there is uncertainty about the non-linear parameter Läuter (1974b) and Chaloner and Larntz (1989) proposed a Bayesian D-optimality criterion which maximizes an expected value of the D-optimality criterion with respect to a prior distribution for the unknown parameter [see also Pronzato and Walter (1985), who called the corresponding designs robust designs, or Chaloner and Verdinelli (1995) for comprehensive reviews of this approach]. Since its introduction Bayesian optimal designs have found considerable attention in the literature [see Haines (1995), Mukhopadhyaya and Haines (1995), Dette and Neugebauer (1997), Han and Chaloner (2004) among others]. Biedermann et al. (2006a) determined efficient designs for binary response models, when there is uncertainty about the form of the link function (e.g. Probit or Logit model) and the parameters. Recently, Woods et al. (2006) used this approach for finding D-optimal designs in the case of uncertainty concerning the parameter vector β as well as the linear predictor $\eta = z'\beta$ and the link function $g(\cdot)$. For this purpose these authors proposed a multi-objective criterion [see Cook and Wong (1994)] for the selection of a design. Most of the optimality criteria in these references are based on the expected value of a given optimality criterion $\Phi(\mathbf{X}|s)$ (such as the D-optimality criterion) over the space $\mathcal{M}$ of the possible models, which takes the model uncertainty into account. In the present context the elements of the set $\mathcal{M}$ are of the form $s = (g, z, \beta)$ corresponding to uncertainty with respect to the link function g, the regression function $z = z(x)$ and the parameter β. To be precise, let $\mathcal{G}$ denote a class of possible link functions. For each $g \in \mathcal{G}$ let $\mathcal{N}_g$ denote a class of vector-valued functions $z(x)$ and finally define for each pair (g, z) with $z \in \mathcal{N}_g$ a parameter space $\mathcal{B}_{g,z}$. With $\mathcal{M} = \{(g, z, \beta) : g \in \mathcal{G}, z \in \mathcal{N}_g, \beta \in \mathcal{B}_{g,z}\}$ the criterion is given by

$$\Phi_B(\mathbf{X}, \mathcal{M}) = \int_{\mathcal{M}} \text{eff}(\mathbf{X}|s) dh_1(\beta|g, z) dh_2(z|g) dh_3(g), \tag{6}$$

where the efficiency is defined by

$$\text{eff}(\mathbf{X}|s) = \Big(\frac{\Phi_D(\mathbf{X}|s)}{\Phi_D(\mathbf{X}_s^*|s)}\Big)^{1/p(s)}, \tag{7}$$

$\mathbf{X}_s^*$ is the locally D-optimal design for model $s \in \mathcal{M}$, $p(s)$ denotes the number of parameters in model s and h_1, h_2 and h_3 represent cumulative distribution functions reflecting the importance of the particular constellation (g, z, β).

As an alternative to the Bayesian criterion Dette (1997) proposed a standardized D-maximin optimality, which determines a design maximizing the worst efficiency over a certain range for the parameter β [see also Müller and Pázman (1998)]. Since its introduction this criterion has found considerable attention in the literature. To be precise, assume that $\mathcal{M}$ is a set of possible values $s = (g, z, \beta)$ for the link function, model and parameter vector and recall the definition of the relative efficiency of the design $\mathbf{X}$ with respect to the locally optimal design $\mathbf{X}_s^*$ defined by (7). The standardized maximin optimal design $\mathbf{X}^*$ is defined as the solution of the optimization problem

$$\max_{\mathbf{X}} \min_{s \in \mathcal{M}} \text{eff}\,(\mathbf{X}|s)\,.$$

Therefore this design maximizes the minimal relative efficiency calculated over the set $\mathcal{M}$, and it can be expected that such a design has reasonable efficiency for any choice of the parameter $s \in \mathcal{M}$.

Standardized maximin optimal designs are extremely difficult to find and for this reason we will mainly consider optimal designs with respect to the Bayesian-type criterion (6). Some explicit results for models with a one-dimensional predictor can be found in Imhof (2001), Dette et al. (2007).

4.3. Design criteria for estimating an additional day-effect

Recall the motivating example discussed at the end of Section 2, where observations are taken at two different days. In order to address this situation in the generalized linear model we replace the regression model $z(x)$ and the parameter β in (1) by the vectors

$$z^*(x,t) = (z(x)^T, t)^T \ ; \quad \beta^* = (\beta^T, \gamma)^T$$

respectively, where the parameter t can attain the values 0 and 1 corresponding to different experimental conditions caused by a possible day effect. Thus the expected response at a particular experimental condition satisfies

$$g(E[Y|x]) = \begin{cases} z^T(x)\beta & \text{if} \quad t = 0 \\ \gamma + z^T(x)\beta & \text{if} \quad t = 1. \end{cases} \tag{8}$$

We assume that n observations are taken at the initial day at experimental conditions $x_1, \dots, x_n$. This corresponds to the choice $t = 0$ and a generalized linear model without the day effect γ is fitted to the data. Additional experiments can be made at any further day at experimental conditions $x_{n+1}, \dots, x_{n+m}$ which corresponds to the choice $t = 1$. Note that in the matrix $\mathbf{X} = (\mathbf{X^{(1)}}, \mathbf{X^{(2)}}) = (x_1, \dots, x_n, x_{n+1}, \dots, x_{n+m})$ the elements in the matrix $\mathbf{X^{(1)}} = (x_1, \dots, x_n)$ are fixed (because they correspond to observations from the initial day) and the criteria are optimized with respect to the experimental conditions $\mathbf{X^{(2)}} = (x_{n+1}, \dots, x_{n+m})$ for the experiments at a different day. We reflect this fact by the notation

$$\Phi_D(\mathbf{X}^{(2)}, s) = \Phi_D((\mathbf{X}^{(1)}, \mathbf{X}^{(2)})) \tag{9}$$

$$\Phi_B(\mathbf{X}^{(2)}, \mathcal{M}) = \Phi_B((\mathbf{X}^{(1)}, \mathbf{X}^{(2)}), \mathcal{M}) \tag{10}$$

$$\text{eff}(\mathbf{X}^{(2)}, \mathcal{M}) \tag{11}$$

for the criteria (5), (6) and the efficiency (7). The corresponding locally optimal designs are denoted by $\mathbf{X}_s^{*(2)}$. Now the question of interest is if the parameter γ vanishes, i.e. if there exists an additional day effect. For this purpose a likelihood ratio test for the hypothesis

$$H_0 : \gamma = 0 \tag{12}$$

on the basis of all $n + m$ observations is performed. The Fisher information for a specific model, weight function (corresponding the generalized linear model) and parameter is then given by

$$I(\mathbf{X}, s) = \sum_{i=1}^{n+m} w(z_i^{*T}\beta^*) z_i^* z_i^{*T} \in \mathbb{R}^{p+1 \times p+1} \tag{13}$$

where $z_i^* = z(x_i, t_i)$ denotes the vector of regression functions corresponding to the i-th observation ($i = 1, \ldots, n+m$) and the weight function is defined by

$$\frac{1}{\left(z_i^{*T}\beta^*\right)^2}, \quad \frac{1}{\left(z_i^{*T}\beta^*\right)^2}, \quad 1$$

for the identity, inverse and log-link, respectively. Standard results on the asymptotic properties of the likelihood ratio test show that the power of the test for the hypothesis (12) in model $s = (g, z, \beta)$ is an increasing function of the quantity

$$\Phi_{D_1}(\mathbf{X}^{(2)}, s) = (e_{p+1}^T I^{-1}(\mathbf{X}, s) e_{p+1})^{-1} \tag{14}$$

where $\mathbf{X} = (\mathbf{X}^{(1)}, \mathbf{X}^{(2)})$, $\mathbf{X}^{(1)} = (x_1, \ldots, x_n)$, $\mathbf{X}^{(2)} = (x_{n+1}, \ldots, x_{n+m})$ and $e_{p+1} = (0, \ldots, 0, 1)^T$ denotes the $(p+1)$-th unit vector in $\mathbb{R}^{p+1}$ [see Dette et al. (2008)]. Consequently, an optimal design for investigating the existence of a day effect if a particular model $s = (g, z, \beta)$ is used for the data analysis maximizes the function $\Phi_{D_1}(\mathbf{X}^{(2)}, s)$ with respect to the choice of the experimental conditions $\mathbf{X}^{(2)} = (x_{n+1}, \ldots, x_{n+m})$ for the m observations taken at any further day. The criterion defined by (14) ist called D_1-optimality criterion in the literature. D_1-optimal designs have been studied by several authors in the context of linear and nonlinear regression models [see Studden (1980), Dette et al. (2005) or Dette et al. (2010) among others], but less work can be found for generalized linear models.

In order to address uncertainty with respect to the model assumptions we denote by $\mathbf{X}_s^{*(2)}$ the locally D_1-optimal design maximizing the criterion defined in (14) and define the D_1-efficiency of a design $\mathbf{X}$ in model $s = (g, z, \beta)$ by

$$\text{eff}_1(\mathbf{X}^{(\mathbf{2})}|s) = \frac{\Phi_{D_1}(\mathbf{X}^{(2)}|s)}{\Phi_{D_1}(\mathbf{X}_s^{*(2)}|s)}. \tag{15}$$

The Bayesian D_1-optimality criterion is finally defined by

$$\Phi_{B_1}(\mathbf{X^{(2)}}, \mathcal{M}) = \int_{\mathcal{M}} \mathrm{eff}_1(\mathbf{X^{(2)}}|s) dh_1(\beta|g,z) dh_2(z|g) dh_3(g) \tag{16}$$

where h_1, h_2 and h_3 represent again cumulative distribution functions reflecting the importance of the particular constellation (g, z, β). Criteria of this type have been discussed by several authors in the case of linear regression models [see Dette (1994), Dette and Haller (1998)].

	Temperature	Velocity	Flame Width	Flame Intensity
Main effects	L, K, D	L, K, D, FDV	L, K, D, FDV	L, K, D, FDV
Squared effects	K^2	K^2	K^2	L^2, K^2, FDV^2
Interaction terms	–	$L \cdot K$	–	$D \cdot FDV$
Link	identity	logistic	inverse	identity
BIC	245.744	196.979	99.749	106.148

Table 4.1: *The generalized linear models chosen by the BIC-criterion for the four responses observed in the thermal spraying process.*

5. Optimal designs for thermal spraying

Recall the problem of designing additional experiments for the thermal spraying described in Section 2. In the application 30 observations have already been made on the basis of a central composite design $\mathbf{X}_R^{(1)}$ (see Table B.1 in Appendix B) while eight additional experiments are conducted for the investigation of an additional day effect. For each response (temperature, velocity, flame width, flame intensity) the data from the first day has been used to identify a generalized linear model in the class of all models with the three link functions specified in Section 3 and different forms for the vector z on the basis of the BIC-criterion. The corresponding results are listed in Table 4.1. For each response the parameter estimates corresponding to the model chosen by the BIC criterion are shown in Tables A.1 - A.4 in Appendix A. For example, for the temperature the BIC criterion selects the generalized linear model with gamma distribution and identity link where the linear part of the model is given by

$$z^T(x)\beta = \beta_0 + \beta_1 L + \beta_2 K + \beta_3 D + \beta_4 K^2.$$

Temperature	Velocity	Flame Width	Flame Intensity
0.6947	0.558	0.5272	0.4508
0.7019	0.5770	0.5904	0.5269
0.6955	0.5741	0.5971	0.5359

Table 5.1: *First row: D-efficiencies of the reference design. Second row: D-efficiencies of the reference design $\mathbf{X}_R$ with respect to the design $\mathbf{X}_B^*$ maximizing the multi objective criterion* (10)*, where γ has been fixed. Third row: D-efficiencies of the reference design $\mathbf{X}_R$ with respect to the design $\mathbf{X}_B^*$ maximizing the multi objective criterion* (10)*, where uncertainty with respect to the parameter γ has been addressed.*

The values of the parameters $(\beta_0, \ldots, \beta_4)$ can be obtained from Table A.1. For the investigation of an additive day effect a reference design $\mathbf{X}_R^{(2)} = (x_{31}, \ldots, x_{38})$ for the eight additional experiments was proposed, which is shown in Table B.2. In order to investigate the efficiency of this design we have calculated the best locally D-optimal designs for the models which were identified by the BIC for modeling the four responses with an additional day effect. These designs require the specification of the unknown parameters and we used the available information from the first 30 experiments of the first day to estimate β (see Tables A.1 - A.4) while the parameter γ for the additional day effect was chosen as $\gamma = -16$, $\gamma = 0.01$, $\gamma = 0.002$ and $\gamma = 0.09$ in the models for temperature, velocity, flame width and flame intensity, respectively.

5.1. D-optimal designs

The corresponding locally D-optimal designs are shown in the Tables B.3 - B.4 in Appendix B, while the corresponding D-efficiencies

$$\mathrm{eff}(\mathbf{X}_R|s) = \Big(\frac{|(\mathbf{X}_R, s)|}{|(\mathbf{X}_s^*, s)|}\Big)^{1/(p(s)+1)}$$

for the designs $\mathbf{X}_R = (\mathbf{X}_C^{(1)}, \mathbf{X}_R^{(2)})$ and $\mathbf{X}_s^* = (\mathbf{X}_C^{(1)}, \mathbf{X}_s^{*(2)})$ are depicted in the first row of Table 5.1 (here $p(s)+1$ denotes the number of parameters in the corresponding model where $p(s)$ parameters appear in regression function $z^T\beta$). We observe that for each type of response the locally D-optimal design yields a substantial improvement of the reference design. The efficiency of the reference design varies between 45% - 70%. Recall that all responses are observed simultaneously. Because the main goal

Temperature	Velocity	Flame Width	Flame Intensity
1.3050	2.2772	1.4284	3.6028
1.3312	1.3534	1.2054	5.4944
1.3121	1.3504	1.2447	5.2482

Table 5.2: *First row: Efficiencies of the reference design with respect to the locally D-optimal designs for estimating the parameter* γ *(see formula* (17)*). Second row: Efficiencies of the reference design* $\mathbf{X}_R$ *with respect to the design* $\mathbf{X}_B^*$ *maximizing the multi objective D-criterion* (10)*, where* γ *has been fixed. Third row: Efficiencies of the reference design* $\mathbf{X}_R$ *with respect to the design* $\mathbf{X}_B^*$ *maximizing the multi objective D-criterion* (10)*, where uncertainty with respect to the parameter* γ *has been addressed.*

of the experiment is to answer the question of additional day effects we display in Table 5.2 the efficiencies

$$\mathrm{eff}_{D_1}(\mathbf{X}_R, \mathbf{X}_s^*) = \frac{\Phi_{D_1}(\mathbf{X}_R^{(2)}|s)}{\Phi_{D_1}(\mathbf{X}_s^{*(2)}|s)} \tag{17}$$

of the reference design with respect to the locally D-optimal design for estimating the parameter γ. The efficiency of the locally D-optimal designs are always larger than 100% compared to the reference design. That means that the locally D-optimal design does not yield an improvement of the reference design when the goal of the experiment is a most precise estimation of the additional day effect. Therefore we also calculate locally D_1-optimal designs in Section 5.2 in order to test for a day effect.

Note that the selected models for the four responses differ and it is not clear if a locally D-optimal design for a particular model (for example the model used for temperature) has good properties in the models used for the other responses. In order to address this problem we have used the multi-objective criterion (10) to find a design $\mathbf{X}^{(2)}$ for the observations on a different day with good efficiencies in all models. We begin considering only uncertainty with respect to the model in the criterion (7), while all the parameters (and link functions) are fixed. We used equal weights for all four models as prior distribution and the resulting design is given in the left part of Table 5.3.

The corresponding efficiencies

$$\mathrm{eff}(\mathbf{X}_R, \mathbf{X}_B^*) = \Big(\frac{|(\mathbf{X}_R, s)|}{|(\mathbf{X}_B^*, s)|}\Big)^{1/(p(s)+1)} \tag{18}$$

Run	L	K	D	FDV	L	K	D	FDV
1	2	-2	-2	-2	2	-2	2	-2
2	2	-2	2	-2	2	-0.15	-2	2
3	2	0.25	2	2	2	0.19	-2	-2
4	2	2	-2	-0.53	2	2	2	2
5	-2	-2	-2	2	-2	-2	-2	-2
6	-2	-2	2	2	-2	-2	2	-0.46
7	-2	0.05	-2	2	-2	0.09	2	2
8	-2	0.2	2	-2	-2	2	-2	2

Table 5.3: *Bayesian D-optimal designs for the four response models. Left part: parameter of the day effect is fixed; right part: three values for the parameter of the day effect, γ, $\gamma \pm 10\%$.*

$$\mathrm{eff}_{D_1}(\mathbf{X}_R, \mathbf{X}_B^*) = \frac{\Phi_{D_1}(\mathbf{X}_R^{(2)}, s)}{\Phi_{D_1}(\mathbf{X}_B^{*(2)}, s)} \tag{19}$$

of the reference design $\mathbf{X}_R$ with respect to the design $\mathbf{X}_B^* = (\mathbf{X}_C^{(1)}, \mathbf{X}_B^{*(2)})$ are presented in the second line of Table 5.1 and 5.2, respectively. We observe a similar improvement as obtained by the locally D-optimal designs for the D-efficiencies. From this table we can easily calculate the D-efficiencies of the design ${\mathbf{X}_B^*}^{(2)}$, which are given by 0.9856, 0.9076, 0.8102, 0.8757 in the models for the temperature, velocity, flame width and flame intensity, respectively. Similarly, the efficiencies $\mathrm{eff}_{D_1}(\mathbf{X}_B^*, \mathbf{X}_s^*)$ of the design ${\mathbf{X}_B^*}^{(2)}$ with respect to the locally D-optimal designs for estimating the paramameter γ are obtained as 0.9803, 1.6825, 1.1850, 0.6557.

While rather precise information is available for the parameter β from the first 30 observations, the designs and its properties might be sensitive with respect to the specification of the parameter γ for the additional day effect. In order to construct designs, which address this uncertainty we can also use the criterion (7), where we now also allow for uncertainty with respect to the parameter γ in the criterion. More precisely, for each of the four models we consider 3 possible values for γ, namely the value used in the locally D-optimality criterion and 90% and 110% of this value (for example for the temperature model we used 14.4, 16, and 17.6 as possible values of γ). The resulting criterion (7) therefore consists of a sum of 12 terms and the maximizing design is depicted in the right part of Table 5.3. The structure of the two Bayesian D-optimal designs is very similar, since both designs put most of the design points in the edges of the design space. The D- and D_1-efficiencies are

Temperature	Velocity	Flame Width	Flame Intensity
0.9933	0.9912	0.9274	0.92598
0.9933	0.9919	0.9323	0.9349
0.9936	0.9946	0.9464	0.9367

Table 5.4: *First row: D_1-efficiencies of the reference design. Second row:Efficiencies of the reference design $\mathbf{X}_R$ with respect to the design $\mathbf{X}^*_{B_1}$ maximizing the multi objective criterion* (16)*, where γ has been fixed. Third row: Efficiencies of the reference design $\mathbf{X}_R$ with respect to the design $\mathbf{X}^*_{B_1}$ maximizing the multi objective criterion* (16)*, where uncertainty with respect to the parameter γ has been addressed.*

presented in the third rows of Table 5.1 and 5.2. Because of the similarity of the two Bayesian D- optimal designs the efficiences have nearly the same values.

These investigations show that the D-optimal designs yield a substantial improvement of the reference design if all parameters in the model (8) have to be estimated. On the other hand, if the only interest of the experiment is the estimation of a day effect, the reference design yields a more precise estimate of the parameter γ than optimal designs based on D-optimality criteria.

5.2. Optimal designs for testing for a day effect

If the main interest of the experiment is the existence of an additional day effect the design can be constructed such that the test for the hypothesis $H_0 : \gamma = 0$ is most powerful, which is reflected by the criterion Φ_{D_1} defined in (14). The corresponding multi-objective criterion addressing uncertainty with respect to the regression model, link function and parameters is given by (16). The locally D_1-optimal designs for the four models in Table 4.1 are presented in right parts of Table B.5 and B.6 in Appendix B, while the efficiency of the reference designs $\mathbf{X}_R$ are given in the first row of Table 5.4. For the temperature and velocity the D_1-efficiencies of the reference design are about 99%. On the other hand an improvement of the reference designs can be observed for the flame width and flame intensity (here the efficiencies are 92.7% and 92.5%, respectively).

As in the previous section we construct a robust design for testing for an additional day effect by maximizing the multi objective criterion (16), where all parameters have been fixed (β is obtained from Tables A.1 - A.4, while information from other experiments was used for the parameter γ, that is $\gamma = -16$, $\gamma = 0.01$, $\gamma = 0.002$ and

Run	L	K	D	FDV	L	K	D	FDV
1	-1.94	0.68	-1.71	2.00	-1.92	0.49	-1.10	2.00
2	-1.16	0.63	1.75	1.27	-1.57	0.19	-0.82	-1.78
3	-0.41	-2.00	0.68	-0.79	0.54	0.13	1.69	-0.25
4	0.16	-0.98	1.86	-0.64	0.22	-2.00	1.06	-0.58
5	0.77	0.10	1.16	-0.52	0.38	-0.73	-1.41	-0.23
6	0.87	0.32	-0.14	-0.50	0.59	0.24	1.54	-0.20
7	1.03	0.54	-1.98	-0.50	0.73	0.52	0.56	-0.02
8	1.03	0.50	-1.45	-0.50	1.17	1.10	-1.41	0.14

Table 5.5: *Bayesian D_1-optimal designs for the four response models. Left part: parameter of the day effect is fixed; right part: three values for the parameter of the day effect, γ, $\gamma \pm 10\%$*

$\gamma = 0.09$ in the models for temperature, velocity, flame width and flame intensity, respectively). The resulting design is shown in the left part of Table 5.5 and its efficiencies are presented in the second row of Table 5.4. We observe a similar improvement of the reference designs as obtained by the locally D_1-optimal designs. Finally, we consider designs addressing the fact that the parameter γ cannot be estimated from the data of the initial day. If we address the uncertainty about this parameter in the same way as described in the previous section we obtain the design presented in the right part of Table 5.5. The efficiencies of the reference designs $\mathbf{X}_R$ with respect to this design are shown in the third row of Table 5.4.
Both Bayesian D_1-optimal designs are very similar but differ substantially from the two Bayesian D-optimal designs in Table 5.3. The D_1-optimal designs put more observations in the interior of the design space $[-2, 2]^4$. Nevertheless their effiencies are very similar and range between 93% and 99%. Whereas the reference design performs nearly as well as the two Bayesian D_1-optimal designs in the cases of temperature and velocity, in the cases of flame width and flame intensity the Bayesian D_1-optimal yields more precise estimates as to the reference designs.

Acknowledgements The authors would like to thank Martina Stein, who typed parts of this manuscript with considerable technical expertise. This work has been supported in part by the Collaborative Research Center "Statistical modeling of non-linear dynamic processes" (SFB 823, Projects B1 and C2) of the German Research Foundation (DFG).

A. Parameters estimates in the identified models

In this section we display the parameter estimates in the models identified by the BIC criterion for the four responses. The values are obtained from the 30 observations of the first day and are used in the local optimality criteria to construct the optimal design for the additional eight runs on the next day.

	Estimate	Std. Error	t value	Pr(> |t|)
(Intercept)	1523.263	2.672	570.036	5.955e-53
L	-17.742	2.314	-7.669	5.035e-08
K	19.658	2.294	8.570	6.554e-09
D	-13.818	2.314	-5.973	3.092e-06
K^2	-9.990	2.081	-4.800	6.259e-05

Table A.1: *Parameter estimates of the model for temperature chosen by the BIC criterion (for this model the link function is the identity function).*

	Estimate	Std. Error	t value	Pr(> |t|)
(Intercept)	6.565e+00	1.564e-03	4198.393	3.510e-69
L	1.361e-02	1.354e-03	10.050	6.962e-10
K	5.161e-02	1.354e-03	38.112	2.732e-22
D	-1.711e-02	1.354e-03	-12.634	7.860e-12
FDV	-7.809e-03	1.354e-03	-5.767	7.112e-06
$L \cdot K$	-3.067e-03	1.659e-03	-1.849	7.733e-02
K^2	-9.169e-03	1.236e-03	-7.417	1.531e-07

Table A.2: *Parameter estimates of the model for velocity chosen by the BIC criterion (for this model the link function is the logistic function).*

	Estimate	Std. Error	t value	Pr(> |t|)
(Intercept)	8.630e-02	1.808e-03	47.727	2.673e-25
L	5.296e-03	1.524e-03	3.476	1.956e-03
K	-4.439e-03	1.612e-03	-2.754	1.106e-02
D	2.867e-03	1.525e-03	1.880	7.236e-02
FDV	-1.231e-02	1.508e-03	-8.162	2.208e-08
K^2	3.904e-03	1.542e-03	2.532	1.832e-02

Table A.3: *Parameter estimates of the model for flame width chosen by the BIC criterion (for this model the link function is the inverse function).*

	Estimate	Std. Error	t value	Pr(> |t|)
(Intercept)	19.478	0.336	57.905	1.188e-24
L	-0.889	0.190	-4.675	1.296e-04
K	0.865	0.186	4.641	1.405e-04
D	-0.371	0.197	-1.883	7.360e-02
FDV	2.166	0.204	10.606	6.851e-10
L^2	-0.310	0.176	-1.759	9.311e-02
K^2	-0.561	0.170	-3.306	3.365e-03
FDV^2	0.509	0.192	2.646	1.512e-02
$D \cdot FDV$	0.410	0.238	1.722	9.976e-02

Table A.4: *Parameter estimates of the model for flame intensity chosen by the BIC criterion (for this model the link function is the identity function).*

B. Appendix: Standard and optimal designs

Run	L	K	D	FDV
1	1	-1	1	-1
2	1	1	1	1
3	-1	-1	1	-1
4	-1	-1	-1	1
5	0	0	0	0
6	0	0	0	0
7	-1	1	1	-1
8	-1	1	-1	1
9	1	1	-1	1
10	1	-1	-1	-1
11	0	0	0	0
12	-1	1	-1	-1
13	1	1	-1	-1
14	-1	1	1	1
15	1	-1	1	1
16	-1	-1	1	1
17	-1	-1	-1	-1
18	1	1	1	-1
19	0	0	0	0
20	1	-1	-1	1
21	0	0	0	0
22	0	0	-2	0
23	-2	0	0	0
24	2	0	0	0
25	0	0	0	0
26	0	0	0	-2
27	0	0	2	0
28	0	2	0	0
29	0	0	0	2
30	0	-2	0	0

Table B.1: *Central Composite Design used for the first* 30 *observations*

Run	L	K	D	FDV
1	1	1	-1	-1
2	1	-1	-1	1
3	-1	-1	-1	-1
4	-1	-1	1	1
5	-1	1	1	-1
6	1	1	1	1
7	1	-1	1	-1
8	-1	1	-1	1

Table B.2: *The reference design for* $\mathbf{X}_R$

	Temperature			Velocity			
Run	L	K	D	L	K	D	FDV
1	-2	-2	2	2	-2	-2	-2
2	-2	-2	2	- 2	-2	2	2
3	-2	0	-2	2	2	2	2
4	-2	2	2	2	2	-2	-2
5	2	-2	-2	-2	-2	-2	-2
6	2	-0.1	-2	2	-0.22	2	-2
7	2	0	2	-2	2	2	-2
8	2	2	2	-2	2	-2	2

Table B.3: *Locally D-optimal designs for the responses temperature (left part) and velocity (right part)*

	Flame Width				Flame Intensity			
Run	L	K	D	FDV	L	K	D	FDV
1	-2	-2	-2	2	2	2	2	-2
2	2	0.49	-2	2	2	-2	2	-0.46
3	-2	0.14	-2	2	2	-2	2	2
4	-2	0.31	-2	-2	0.33	-0.31	2	-2
5	-2	2	-2	2	-2	-2	2	-2
6	-2	2	2	2	1.35	-2	-2	2
7	2	-2	2	2	2	-2	2	-2
8	-2	0.17	2	2	2	-2	-2	-2

Table B.4: *Locally D-optimal designs for the responses flame width (left part) and flame intensity (right part)*

Run	L	K	D	L	K	D	FDV
1	0.11	-0.12	0.55	0.70	0.25	-1.20	-0.07
2	-1.75	-1.11	-1.32	-0.79	-1.88	0.06	0.19
3	0.67	0.08	1.60	1.80	0.35	0.77	-1.05
4	0.39	0.41	1.20	-0.98	0.94	-1.18	1.13
5	-0.36	1.35	-1.02	-0.47	-0.06	1.54	-1.25
6	0.34	0.04	-1.18	0.13	-0.46	-0.36	1.11
7	0.67	-1.72	-0.61	-0.94	1.20	-0.71	1.67
8	0.29	0.66	1.04	0.55	-0.34	1.04	-1.72

Table B.5: *Locally D_1-optimal designs for the responses temperature (left part) and velocity (right part)*

Run	L	K	D	FDV	L	K	D	FDV
1	1.33	0.16	-0.36	-1.76	0.44	-1.24	0.84	-1.98
2	-1.11	0.40	-1.75	1.83	-1.33	0.00	0.38	-0.42
3	1.63	-1.28	0.24	1.92	0.70	0.45	-0.35	-0.40
4	-1.43	1.05	1.21	-0.59	0.32	-1.53	-.077	0.41
5	-1.54	-0.93	0.09	0.81	-0.62	-0.97	1.63	0.29
6	1.86	0.81	-1.78	-0.92	1.34	0.11	0.17	-0.22
7	-1.53	0.95	1.71	1.53	1.21	0.79	-1.66	-0.20
8	0.78	0.03	-0.99	-1.85	-0.91	1.36	0.95	0.27

Table B.6: *Locally D_1-optimal designs for the responses flame width (left part) and flame intensity (right part)*

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